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STOCHASTIC DYNAMIC FINITE ELEMENT ANALYSIS OF A NONUNIFORM BEAM

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Abstract—In this paper, the statistical dynamic responses of a nonuniform beam with stochastic Young's modulus of elasticity are obtained by using the perturbation technique in conjunction with the finite element method. The proposed method produces the expectation, variance and autocorrelation function of the deflection, strain and stress that are quite useful in estimating the structural safety and reliability. Some statistical responses obtained by the perturbation method are checked by the Monte Carlo simulation, also, the reliability analysis of the structure is performed based on certain failure criteria of the structure.

1. INTRODUCTION

Since the early 1970s, the applications of the stochastic finite element method have been developed rapidly. Among others, the following researchers have already contributed to the development of this field. Astill *et al.* (1972) used the Monte Carlo simulation method to study the structural response variability due to random properties in structures. Hisada and Nakagiri (1980a, b, 1981, 1983, 1985) used the perturbation method with finite element method to investigate some structural problems. The stochastic finite element method was adopted by Baecher and Ingra (1981) to predict the settlement of the structure on foundation. Vanmarcke and Grigoriu (1983) also used the stochastic finite element method to demonstrate the use of spatial averaging of random fields to simple beams with random rigidity. A similar approach has been used by Liu *et al.* (1986) with applications to elasto-plastic and nonlinear dynamic problems. Recently, Shinozuka and Dasgupta (1986) have adopted Neumann expansion with Monte Carlo simulation to demonstrate the advantages of the formulation to dynamic problems. Also, Der Kiureghian and Ke (1988) used the stochastic finite element method to study the safety and reliability of the structure. Shinozuka and Dasgupta (1986) and Shinozuka and Deodatis (1988) used the finite element method in conjunction with Neumann expansion and Monte Carlo simulation techniques to investigate the statistical dynamic response of the structure due to random material properties or geometries in structures and studied the structural safety and reliability. More recently, Bucher and Shinozuka (1988) and Kardara *et al.* (1989) used the Green's function formulation to determine the mean square response of statistically indeterminate structures.

In general, the structural uncertainties might include the Young's modulus of elasticity, Poisson's ratio, cross section, length and the geometric imperfection of the beam. However, in this study, only the Young's modulus of elasticity is considered as a stochastic field with respect to the position, and then the stochastic finite element method along with perturbation method is used to investigate the stochastic dynamic response of a nonuniform beam which is loaded with a deterministic transverse dynamic load. These statistical dynamic responses can be used for estimating the reliability of the structure.

2. DYNAMIC FINITE ELEMENT EQUATION

Considering a simple beam element with uniform cross section area A, constant moment of inertia I, Young's modulus of elasticity E, density ρ and length l. The simple beam element is assumed to have two degrees of freedom at each nodal point: a transverse deflection w and an angle of rotation θ (or $\partial w/\partial x$). To derive dynamic finite element equation of motion of the beam, we use the Lagrange's equation with the strain energy and kinetic energy as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right)+\left(\frac{\partial U}{\partial q_i}\right) = F_i, \tag{1}$$

where T is the kinetic energy of a beam element, U is the strain energy of a beam element, both energies are functions of the translational displacement w, also F_i is the nodal force or moment, q_i is the nodal translational displacement or rotational displacement, subscript i is the degree of freedom number. Substituting the displacement function into the expressions for both T and U and then performing the differentiation as indicated in eqn (1), we obtain:

$$F_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{w}}\right)+\frac{\partial U}{\partial w} = \begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \end{bmatrix}^t \begin{bmatrix} \ddot{w}_1 \\ \ddot{\theta}_1 \\ \ddot{w}_2 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_{11} \\ k_{12} \\ k_{13} \\ k_{14} \end{bmatrix}^t \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}, \tag{2}$$

where w_i's and θ_i's are the translational displacements and rotational displacements at the two ends of the beam element, respectively. In general, the m_{ij} and k_{ij} in eqn (2) can be expressed in the following form individually,

$$m_{ij} = \rho A \int_0^l \phi_i \phi_j \,\mathrm{d}x, \tag{3}$$

$$k_{ij} = EI \int_0^l \phi_i'' \phi_j'' \,\mathrm{d}x, \tag{4}$$

where ϕ_i's are the shape functions representing the deflection curve for the beam element produced by setting the corresponding degree of freedom to one and the remaining degrees of freedom to zero. Based on eqns (3) and (4) it is quite straightforward to obtain the local consistent mass matrix $[m]$ and local stiffness matrix $[k]$. Assembling each element and considering the damping forces, we can derive a global dynamic finite element equation of motion for a nonuniform beam as follows:

$$[M][\ddot{W}]+[C][\dot{W}]+[K][W] = [F], \tag{5}$$

where $[M]$ is the global consistent mass matrix, $[C]$ is the global damping matrix, $[K]$ is the global stiffness matrix, $[F]$ is the global forcing vector involving the transverse distributed loading $p(x, t)$. The introduction of boundary conditions into eqn (5) is of course done in the standard way.

3. PERTURBATION TECHNIQUE

In this study, only the Young's modulus of elasticity E is considered to be stochastic in position, the moment of inertia, the geometric shapes and sizes of the structure are assumed to be deterministic as well as the time-dependent distributed load. Applying the perturbation technique, the stochastic Young's modulus can be assumed as:

$$E(x) = E_0(1+\eta(x)) = E_0+E_0\eta(x), \tag{6}$$

where E_0 is the mean value of the modulus of elasticity and η is a zero-mean, one dimensional real homogeneous stochastic field, denoting the fluctuation of the elastic modulus around its mean value and is assumed to be uniform within each element. Then the stochastic local and the global matrices can be written individually as follows:

$$[k] = [k^0]+[k^1]\eta(x), \tag{7}$$

$$[K] = [K^0]+\sum_{i=1}^{NE}[K_i^1]\eta_i, \tag{8}$$

where the superscript 0 represents the deterministic term, and Σ is the merging with respect to the element, i means the element number and NE is the total number of the element. Similarly, the nodal displacement vector [**W**] can be assumed as:

$$[\mathbf{W}] = [\mathbf{W}^0]+\sum_{i=1}^{NE}[\mathbf{W}_i^1]\eta_i+\sum_{i=1}^{NE}\sum_{j=1}^{NE}[\mathbf{W}_{ij}^2]\eta_i\eta_j, \tag{9}$$

where j is the element number as well as i counting the number of elements.

Substituting eqns (8)–(9) into (5) and applying the second order perturbation technique to eqn (5), we obtain the following:

$$[M][\ddot{\mathbf{W}}^0]+[C][\dot{\mathbf{W}}^0]+[K^0][\mathbf{W}^0] = [F], \tag{10}$$

$$[M][\ddot{\mathbf{W}}_i^1]+[C][\dot{\mathbf{W}}_i^1]+[K^0][\mathbf{W}_i^1] = -[K_i^1][\mathbf{W}_0], \tag{11}$$

$$[M][\ddot{\mathbf{W}}_{ij}^2]+[C][\dot{\mathbf{W}}_{ij}^2]+[K^0][\mathbf{W}_{ij}^2] = -[K_i^1][\mathbf{W}_j]. \tag{12}$$

The solutions of eqns (10)–(12) are achieved by using a step by step numerical integration algorithm. More specifically, the Newmark integration method is chosen to solve $[\mathbf{W}^0]$, $[\mathbf{W}_i^1]$, and $[\mathbf{W}_{ij}^2]$ successively.

4. STATISTICAL DYNAMIC ANALYSIS

The statistical dynamic response of a nonuniform beam can be calculated once $[\mathbf{W}^0]$, $[\mathbf{W}_i^1]$, and $[\mathbf{W}_{ij}^2]$ are obtained. For any fixed time, the autocorrelation of the deflection **W** between two different points p and q can be evaluated by using the first rate of change in nodal displacement $\mathbf{W}_i^1$ as follows:

$$\begin{aligned} R_w(x_p,x_q) &= E\left[\sum_{i=1}^{NE}[\mathbf{W}_{pi}^i]\eta_i\sum_{j=1}^{NE}[\mathbf{W}_{qj}^1]\eta_j\right] \\ &= \sum_{i=1}^{NE}\sum_{j=1}^{NE}[\mathbf{W}_{pi}^1][\mathbf{W}_{qj}^1]E[\eta_i\eta_j], \end{aligned} \tag{13}$$

$$\begin{aligned} E[\eta_i\eta_j] &= R_\eta(x_i-x_j) \\ &= R_\eta(\Delta x), \end{aligned} \tag{14}$$

where $E[\cdot]$ is the expectation, and $R_\eta(\Delta x)$ is the autocorrelation of random variable η. If the stochastic process of the random variable η is homogeneous with respect to position, x_p and x_q are the coordinates at the center of the beam elements p and q. When the stochastic process of the settlement w is homogeneous along the x-axis, $R_w(x_p,x_q)$ is replaced by $R_w(x_p-x_q)$, therefore, the autocorrelation $R_w(\Delta x)$ can be computed readily provided that the spectral density $S_\eta(\kappa)$ of random variable η is known. In this paper, the spectral density $S_\eta(\kappa)$ is assumed to be:

$$S_\eta(\kappa) = \frac{1}{2\sqrt{2}}\sigma_\eta^2\exp\left[-\frac{\kappa^2}{2}\right], \tag{15}$$

where σ_η denotes the standard deviation of the stochastic field $\eta(x)$ or the coefficient of variation of the Young's modulus of elasticity, and κ is the wave length. Also, the autocorrelation $R_\eta(\Delta x)$, which is the Wiener–Khintchine transform of the spectral density $S_\eta(\kappa)$, can be expressed as follows:

$$R_\eta(\Delta x) = \sigma_\eta^2 \exp[-(\Delta x)^2]. \tag{16}$$

Therefore, the autocorrelation $R_w(\Delta x)$ can be computed readily from eqns (13)–(16). Meanwhile, the variance vector of the deflection indeed is the diagonal components of eqn (13) as follows:

$$\mathrm{Var}\,[\mathbf{W}] = \sum_{i=1}^{NE} \sum_{j=1}^{NE} \mathrm{diag}\,\{[\mathbf{W}_i^1][\mathbf{W}_j^1]E[\eta_i\eta_j]\}. \tag{17}$$

It should be noted that not only the expectation, variance and autocorrelation $R_w(\Delta x)$ of the deflection w can be obtained but also the statistical strain and stress can be evaluated by a similar procedure. Furthermore, it is also quite straightforward to compute the statistical quantities of the velocity and acceleration.

5. SOME OTHER STATISTICAL RESPONSES

The statistical responses such as strain and stress other than deflection can be calculated by utilizing the standard finite element technique. Noting that the strain-displacement relationship can be described as:

$$[\boldsymbol{\varepsilon}_e] = [B_e][\mathbf{W}_e], \tag{18}$$

where e denotes the eth element and $[B_e]$ is given by the following expression:

$$[B_e] = \left[\frac{\mathrm{d}^2\phi_1}{\mathrm{d}x^2}, \frac{\mathrm{d}^2\phi_2}{\mathrm{d}x^2}, \frac{\mathrm{d}^2\phi_3}{\mathrm{d}x^2}, \frac{\mathrm{d}^2\phi_4}{\mathrm{d}x^2}\right], \tag{19}$$

where ϕ_i's are the shape functions denoted before, and $[\mathbf{W}_e]$ can be written as:

$$[\mathbf{W}_e] = [\mathbf{W}_e^0] + \sum_{i=1}^{NE} [\mathbf{W}_{ei}^1\eta_i] + \sum_{i=1}^{NE} \sum_{j=1}^{NE} [\mathbf{W}_{eij}^2\eta_i\eta_j], \tag{20}$$

then the strain vector of the eth element $[\boldsymbol{\varepsilon}_e] = [\boldsymbol{\varepsilon}_{xx}]$ is obtained by substituting eqn (20) into eqn (18) as follows:

$$[\boldsymbol{\varepsilon}_e] = [\boldsymbol{\varepsilon}_e^0] + \sum_{i=1}^{NE} [\boldsymbol{\varepsilon}_{ei}^1]\eta_i + \sum_{i=1}^{NE} \sum_{j=1}^{NE} [\boldsymbol{\varepsilon}_{eij}^2]\eta_i\eta_j, \tag{21}$$

with the relationship

$$[\boldsymbol{\varepsilon}_e^0] = [B_e][\mathbf{W}_e^0], \tag{22}$$

$$[\boldsymbol{\varepsilon}_{ei}^1] = [B_e][\mathbf{W}_{ei}^1], \tag{23}$$

$$[\boldsymbol{\varepsilon}_{eij}^2] = [B_e][\mathbf{W}_{eij}^2]. \tag{24}$$

Similarly, the stress vector of the eth element $[\boldsymbol{\sigma}_e] = [\boldsymbol{\sigma}_{xx}]$, is represented by stress–strain relationship:

$$[\boldsymbol{\sigma}_e] = [E_e][\boldsymbol{\varepsilon}_e], \tag{25}$$

where $[E_e]$ is the elasticity matrix of the eth element and can be expanded into the following series:

$$[E_e] = [E_e^0]+[E_e^1]\eta_e+[E_{ee}^2]\eta_e^2+\cdots, \tag{26}$$

where $[E_e^0]$ is the elasticity modulus evaluated at $\eta_e = 0$, $[E_e^1]$ and $[E_{ee}^2]$ are the first and second derivatives of $[E_e]$ with respect to η_e and evaluated at $\eta_e = 0$.

Introducing eqns (21) and (26) into eqn (25), $[\sigma_e]$ is obtained as:

$$\begin{aligned}[\sigma_e] &= [[E_e^0]+[E_e^1]\eta_e+[E_{ee}^2]\eta_{ee}^2+\cdots]\left[[\varepsilon_e^0]+\sum_{i=1}^{NE}[\varepsilon_{ei}^1]\eta_i+\sum_{i=1}^{NE}\sum_{j=1}^{NE}[\varepsilon_{eij}^2]\eta_i\eta_j+\cdots\right] \\ &= [\sigma_e^0]+\sum_{i=1}^{NE}[\sigma_{ei}^1]\eta_i+\sum_{i=1}^{NE}\sum_{j=1}^{NE}[\sigma_{eij}^2]\eta_i\eta_j+\cdots,\end{aligned} \tag{27}$$

with the relationships:

$$[\sigma_e^0] = [E_e^0][\varepsilon_e^0], \tag{28}$$

$$[\sigma_{ei}^1] = [E_e^0][\varepsilon_{ei}^1]+\delta_{ie}[E_e^1][\varepsilon_e^0], \tag{29}$$

$$[\sigma_{eij}^2] = [E_e^0][\varepsilon_{eij}^2]+2\delta_{je}[E_e^1][\varepsilon_{ei}^1]+\delta_{ie}\delta_{je}[E_{ee}^2][\varepsilon_e^0], \tag{30}$$

in which δ_{mn} denotes Kronecker's delta.

6. MONTE CARLO SIMULATION

In general, Monte Carlo simulation is used to check the statistical quantities by some other methods, and is very costly since it needs tremendous repeated computations for all different samples, particularly when finite element analysis is adopted. In this paper, the statistical dynamic responses obtained by the perturbation method are checked by Monte Carlo simulation. Since the Young's modulus of elasticity E is assumed as stochastic in position, we have to generate many different sample functions to describe the fluctuation of the random variable η. This can be done by using the following equation:

$$\eta(x) = \sqrt{2}\sum_{i=1}^{N}[S_\eta(\kappa_i)\Delta\kappa]^{1/2}\cos(\kappa_i x-\phi_i), \tag{31}$$

$$\kappa_i = \kappa_l+(i-\tfrac{1}{2})\Delta\kappa, \quad \Delta\kappa = (\kappa_u-\kappa_l)/N, \tag{32}$$

where $S_\eta(\kappa)$ is the one-sided spectral density of the random variable η, κ_u is the upper cutoff wave number of the spectral density, κ_l is the lower cutoff wave number of the spectral density, N is the total number of intervals in the discretization of the spectrum and ϕ_i is an independent random phase angle uniformly distributed between 0 and 2π. Many different sample functions for $\eta(x)$ can be generated from eqn (31), then different sample functions will produce different Young's modulus E, different element stiffness matrix $[k]$ and different global stiffness matrix $[K]$. Then the dynamic finite element equation of motion is solved many times due to many different sample functions using the technique of Newmark integration algorithm. The statistical dynamic response can be computed from all these different dynamic responses.

7. PROBABILITY PAPER AND RELIABILITY ANALYSIS

In the previous sections, we have discussed how to compute some statistical properties of deflection, strain, and stress. However, it is interesting to know what distribution of strain or stress is. There are some methods that can accomplish the above task. In this paper, the so called probability paper is adopted for determining the distribution of the dynamic response. The concept of the probability paper is that a distribution function is represented by a straight line on its associated probability paper. That is, if the distribution function associated with the probability paper is really the population distribution function from which the random processes are taken, then it is expected that these points are scattered around a straight line within a reasonably small deviation. Therefore, the distribution function associated with probability paper is accepted as the population distribution unless those points are evidently scattered around a curved line instead of a straight line. In general, four types of more familiar distribution, specifically the normal, log-normal, asymptotic of the first kind and Weibull are considered in order to find the best possible fit to the data.

As long as the distribution of the statistical response is known, the reliability analysis can be performed. Of course, from the engineering point of view, any of the responses such as deflection, strain or stress can be used as the criteria of the reliability depending on which field of engineering structure and which specification or code we are dealing with. For example, if the stress is used as the criteria of the reliability analysis, and the maximum induced stress is determined to be log-normal distributed due to probability paper plot, then we can define the reliability of the beam in such a way that the beam would survive as long as its maximum induced stress is absolutely less than its allowable stress which is also assumed to be log-normal distributed, then the reliability of the beam is only related to the mean value and the standard deviation of the maximum induced stress and the allowable stress. Then the reliability or safety index β can be defined as follows:

$$\beta = \frac{\ln(\mu_A/\mu_I)}{\sqrt{V_A^2 + V_I^2}}, \tag{33}$$

where μ_A and μ_I are the mean values of the allowable stress and maximum induced stress individually, V_A and V_I are the coefficient of variation of the allowable stress and the maximum induced stress, respectively. The reliability of the beam can then be computed as follows:

$$P_r = \Phi_0(\beta), \tag{34}$$

where Φ_0 is the standardized normal distribution function. Incidentally, the probability of failure of the beam can be expressed as:

$$P_f = 1 - \Phi_0(\beta) = \Phi_0(-\beta). \tag{35}$$

8. NUMERICAL EXAMPLES

In this numerical analysis, the boundary conditions of the beam are assumed as simply supported at both ends. The following parameter values are used for describing the material and geometry of the beam: $L = 100$ ft (30.48 m), $E_0 = 3.0 \times 10^4$ ksi (2.068×10^{11} N m^{-2}), $I_0 = 1000$ in^4 (4.16×10^{-4} m^4), $A_0 = 120$ in^2 (0.0774 m^2), $\sigma_n = 0.1$, $\rho = 15.2$ slug ft^{-3} (7840.2 kg m^{-3}), and

$$A(x) = A_0\left[\frac{2(1-\gamma)}{L}|x - L/2| + \gamma\right], \tag{36}$$

$$I(x) = I_0\left[\frac{2(1-\gamma)}{L}|x - L/2| + \gamma\right]^3, \tag{37}$$

where E_0 is the mean value of the Young's modulus of the beam, I_0 is the moment of inertia of the beam at both ends, A_0 is the area of the cross section of the beam at both ends, γ is a constant to describe the nonuniform property of the beam, γ must be greater than zero in both eqns (36) and (37), and $\gamma = 1.0$ represents a beam with uniform cross section, γ is assumed as 1.1 in the present numerical example. Also, ρ is the density of the beam, and the total numbers of finite element NE is 40. Incidentally, the damping matrix $\mathbf{C}$ of eqn (5) is assumed to be equal to $2\xi\omega_1\mathbf{M}$, where ξ is the damping ratio and ω_1 is the fundamental natural frequency of the nonuniform beam. In this numerical example, $\xi = 0.04$ and $\omega_1 = 4.278943$ rad sec^{-1} that is computed by using the Finite Element Software Package MSC/NASTRAN. The dynamic loading $p(x, t)$ of the beam is assumed to be $p_0p_1(t)$, having a uniform magnitude equal to $p_0 = 10$ lb ft^{-1} (146.1 N m^{-1}), while the function $p_1(t)$ describing its time dependence, which is a rectangular excitation from 0.0–2.0 s with uniform magnitude one. A computer code based on the perturbation method and Monte Carlo simulation method has been written on HP 835/SRX to perform numerical analysis for the problem. To perform Monte Carlo simulation, 250 different samples for the random variable η are generated using the method mentioned previously. In Figs 1, 2, the mean value and standard deviation of the maximum deflection of the beam at 0.75 s are presented along the position. As it can be seen from Figs 1, 2, the results based on the perturbation method and those computed by means of the Monte Carlo simulation show a good agreement. It should be emphasized that the perturbation method is much faster than Monte Carlo simulation as far as the computation time is concerned. The mean value and standard deviation of the strain and stress for two different time instants are plotted in Figs 3–6, respectively. To investigate how the statistical dynamic response change against the time, Fig. 7 shows the mean value of deflection with respect to time for two different damping ratios. To study the distribution of the statistical response, the goodness of fit is tested for some cases such as normal, log-normal, asymptotic of the first kind and Weibull distributions, by plotting the data on the corresponding probability papers. In Fig. 8, based on Monte Carlo simulation analysis, 250 different values of maximum strain of the beam are plotted on the log-normal probability paper, it can be concluded from Fig. 8 that the lognormal distribution fits best for the maximum strain of the beam. To perform the reliability analysis of the beam, the strain and stress criteria are chosen even any other criteria such as deflection could be used. The allowable strain is assumed to be log-normal distributed, and the mean value and the coefficient of variation of the allowable strain are assumed as 0.0003 and 0.2, respectively. Meanwhile, the allowable stress is assumed to be log-normal distributed, and the mean value and the coefficient of variation of the allowable stress are assumed as 0.04% of the mean value of the Young's modulus E_0 and 0.2, respectively. In Fig. 9, the probability of failure of the beam is plotted for both strain and stress criteria with respect to the mean value of the Young's modulus of elasticity, it is quite reasonable that the probability of failure of the beam decreases as the mean value of the Young's modulus of elasticity gets larger.

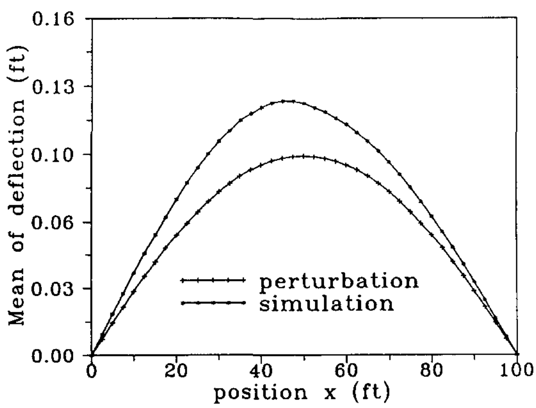

Fig. 1. Mean value of deflection.

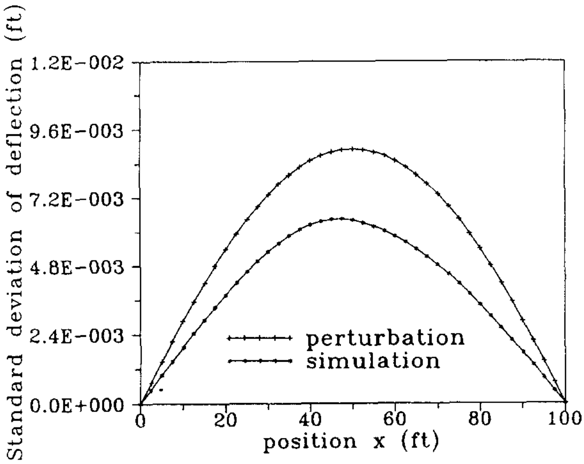

Fig. 2. Standard deviation of deflection.

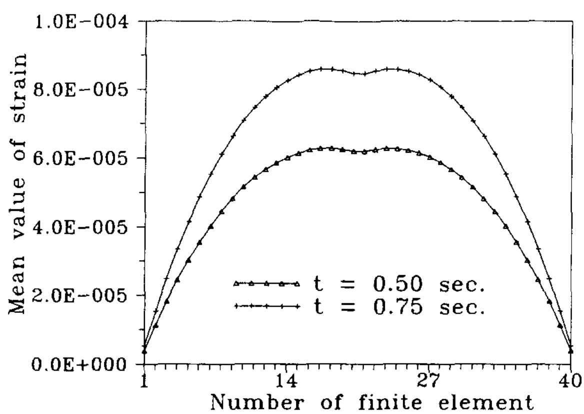

Fig. 3. Mean value of strain.

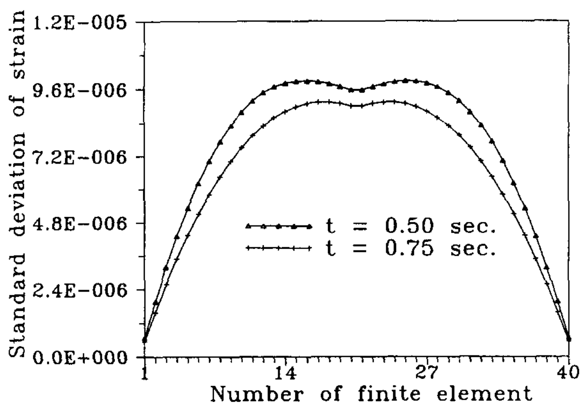

Fig. 4. Standard deviation of strain.

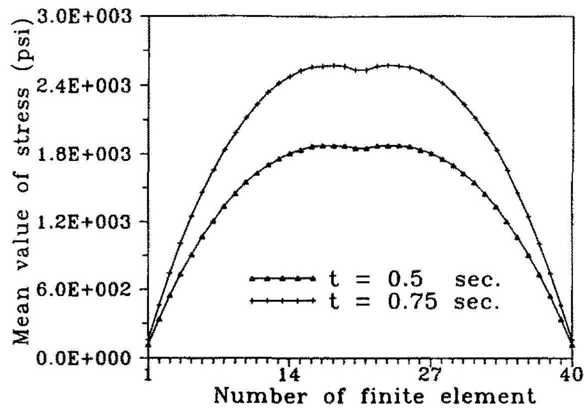

Fig. 5. Mean value of stress.

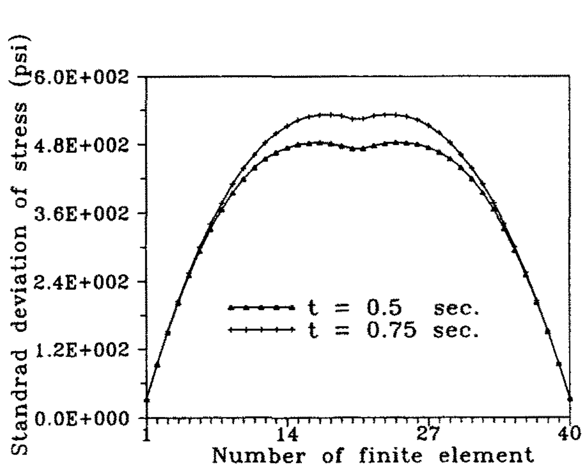

Fig. 6. Standard deviation of stress.

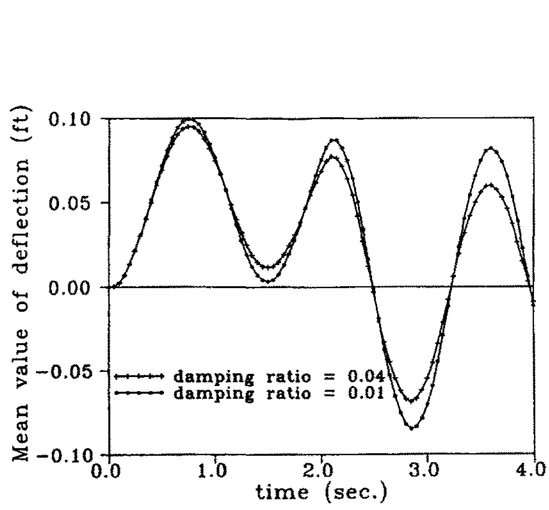

Fig. 7. Mean value of deflection.

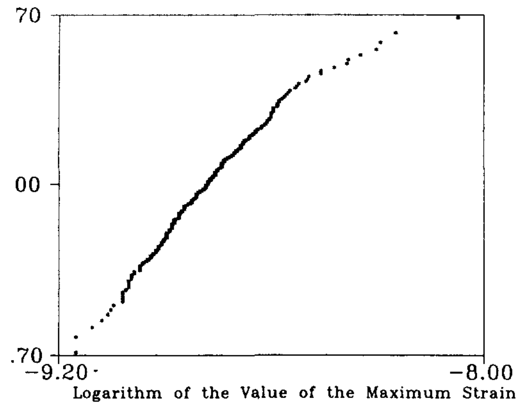

Fig. 8. Two hundred and fifty values for the maximum strain plotted on the log-normal probability paper.

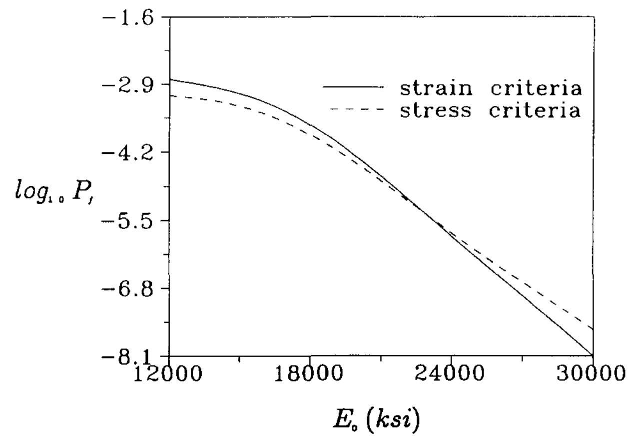

Fig. 9. Probability of failure based on different criteria.

9. SUMMARY

The statistical dynamic responses of a nonuniform beam with stochastic Young's modulus are obtained by using the perturbation technique in conjunction with the finite element method. The proposed method produces the expectation, variance and autocorrelation function of the deflection, strain and stress that are quite useful in estimating the structural safety and reliability. Some statistical responses obtained by the perturbation method are checked by the Monte Carlo simulation, also, the reliability analysis of the structure is performed based on certain failure criteria of the structure.

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